Mach Disk from Underexpanded Axisymmetric Nozzle Flow

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The flowfield associated with the underexpanded axisymmetric nozzle freejet flow including the appearance of a Mach disk has been studied. It is shown that the location and size of the Mach disk are governed by the appearance of a triple-point shock configuration and the condition that the central core flow will reach a state of "choking at a throat." It is recognized that coalescence of waves requires special attention and the reflected wave, as well as the vorticity generated from these wave interactions, have to be taken accurately into account. The theoretical results obtained agreed well with the experimental data.

Nomenclature

c_v = specific heat at constant volume
M = Mach number
M^* = Mach number based on speed of sound at critical condition
N = number of divisions on the starting characteristic line
p = pressure
r = radial coordinate
s = entropy
x = horizontal coordinate
α = Mach angle
γ = ratio of specific heats (= 1.4 here for air)
θ = flow angle

Subscripts

a = ambient condition
c = centerline of symmetry
e = nozzle exit
i = incident wave
l = lower side of slipline
md = Mach disk
o = stagnation state
r = reflected shock wave
sk = imbedded shock wave
u = upper side of slipline
I, II = characteristic of family I or family II

Introduction

FOR a gas jet exhausting from an axisymmetric nozzle to a lower pressure surroundings, the flow will follow a Prandtl-Meyer expansion at the corner of the nozzle. These expansion waves will eventually be reflected as compression waves from the constant pressure jet boundary. When the ratio of exit-to-ambient pressure is high, the compression waves of the same family will unavoidably intersect each other, and shock waves would start to appear as a result of coalescence of these waves. Further downstream of the flow, the snowballing compressive effect will result in an imbedded shock wave which is so strong and curved that regular reflection from the centerline of symmetry is impossible. The appearance of Mach configuration of shock will occur.

Figure 1 shows a typical flow configuration associated with such a freejet flow. The incident shock for the Mach configuration is not straight. The flows upstream of the incident, reflected, and the Mach shocks are nonuniform. Many investigations have been carried out to study the structure of the jet. Pack[2] examined the formation of shock waves in a two-dimensional gas jet by the method of characteristics. Love et al.[3] made an extensive investigation on the axisymmetric jets exhausting from sonic and supersonic nozzles into still air and into supersonic streams. Some theoretical calculations were also presented in their report; however, their study of the Mach configuration was incomplete. Experimental investigation has also been performed by Ladenburg et al.[4] using interferometric study for the axisymmetric gas jet from an orifice instead of a well designed nozzle. Adamson and Nicholls[5] employed a fictitious one-dimensional nozzle extension to find the location of the first Mach shock. Their analysis is very simple; however, it does not yield detailed information of the over-all flowfield. Eastman and Radtke[6] employed a different assumption that the location of the normal Mach shock wave coincided with the point of "minimum" pressure behind the imbedded shock wave.

The criterion of sonic condition occurring at the section of minimum area was used earlier by Chow and Addy[7] in the study of the mutual interactions between primary and secondary streams of a supersonic ejector system. A similar idea has been adopted by Ashratov[8] to find the Mach disk radius of a jet leaving an overexpanded axisymmetric nozzle flow. Recently, this same idea has been suggested by Abbett[9] for the appearance of the Mach disk in underexpanded exhaust plumes. No extensive results of calculations were presented.

It should be recognized that in early studies of the appearance of Mach shock configurations by Bleakney, Taub, and Sternberg[10,11] there exists a regime for weak shock waves that the experimental data do not agree with the predictions from the inviscid flow calculation. Furthermore, the shock waves are strongly curved near the triple point (for subsonic flow behind the reflected shock) that inviscid treatment of the flowfield is inadequate to explain the occurrence of the Mach configuration. In addition, for two-dimensional nozzle freejet flows where the

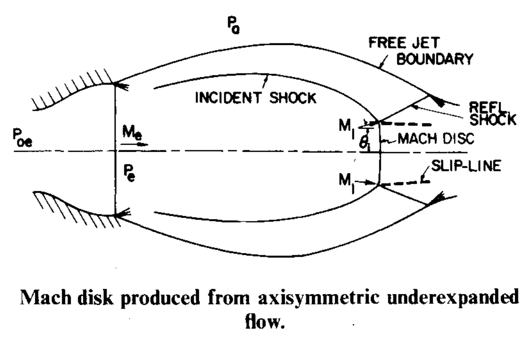

Fig. 1 Mach disk produced from axisymmetric underexpanded nozzle flow.

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Index categories: Shock Waves and Detonation; Jets, Wakes, and Viscid-Inviscid Flow Interactions; Supersonic and Hypersonic Flows.

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Mach configuration appears,[12] the interaction between the outer stream and the central core flow is such that a simple one-dimensional description for the latter is impossible to match with the former in the early portion of the flow development. Nevertheless, in view of the success reported in the investigations mentioned previously, it is believed that the location and the size of the Mach disk associated with underexpanded axisymmetric freejet flows can be considered as a result of the inviscid interaction between the shock waves and the flowfields that an equivalent "choking with a throat" phenomenon occurs somewhere downstream. Indeed, as a continued effort of the study for two-dimensional flow problems,[12] it is the intention of this paper to discuss the special feature for this axisymmetric flow problem including the interaction between the streams, the coalescence of the shock waves, and difficulties encountered in the calculations of these flowfields.

Method of Calculations

Since the major part of the flowfield is supersonic, the method of characteristics for axially symmetrical flows is readily employed. The characteristic equations are given in physical and hodographical planes, respectively, by

$$\left.\frac{dr}{dx}\right|_{\mathrm{I,II}} = \tan(\theta \mp \alpha) \tag{1}$$

$$\left.\frac{1}{M^*}\frac{dM^*}{d\theta}\right|_{\mathrm{I,II}} = \mp\tan\alpha - \left.\frac{\sin^2\alpha}{\gamma(\gamma-1)}\frac{d(s/c_v)}{d\theta}\right|_{\mathrm{I,II}} + \left.\frac{\tan^2\alpha\tan\theta}{\tan\theta\mp\tan\alpha}\frac{1}{r}\frac{dr}{d\theta}\right|_{\mathrm{I,II}} \tag{2}$$

The second term of the right-hand side of Eq. (2) represents the influence of vorticity generated from curved shocks. In many of the flow situations, where the influence of vorticity is relatively unimportant, ignoring this effect would simplify the calculation procedure considerably. However, for the present problem, it will be seen that the vorticity effect is very important and must be accurately taken into account. Application of Eqs. (1) and (2) in finite-difference form to the flowfield is well known and will not be repeated here.

A. Wave Coalescence

For a certain flow condition, the characteristic curves of the same family are found to intersect each other in the physical plane during the process of computation. It indicates the failure of the continuous, single-valued solution, and a shock discontinuity should accordingly be introduced into the flowfield. This is where the imbedded shock wave originates. The weak reflected wave resulting from the intersection of the characteristic compression wave and shock wave should be taken into account since it will further be reflected from the freejet boundary and has the augmenting effect on the downstream shock strength. In this Mach disk problem, the accurate calculation of incident shock strength is found to be critical since all the downstream flowfield, including the triple-point solution, depends on it. Therefore, the region where compression wave and shock wave of the same family intersect to produce a stronger shock wave should be examined carefully and is of paramount importance. Moe and Troesch[13] are correct in stating that the calculation of the imbedded shock wave is the most complicated part of the numerical method. The simple fold-back method[3] or other simplified methods, though determining *the shock position* correctly in the early part of the flowfield, will not give the appropriate shock wave strength.

The arrangement of the calculations to account for this phenomenon of coalescence is reported in detail,[1] and will not be presented here.

B. Triple Point

The Mach configuration is characterized by the interaction between several shock waves. This distinctive feature is manifested through the appearance of a triple point where shocks of different strengths intersect each other. The flow condition in the neighborhood of this triple point has the dominant influence on the downstream flowfield because of the existence of subsonic flow behind the Mach shock. As mentioned in the introduction, the strengths of reflected and Mach shocks in the immediate vicinity of triple point can be determined from the inviscid considerations that the pressure is continuous and the flow angles are the same across the slipline. Since these relations are well known, the presentation of the detailed triple-point solutions is omitted.

C. Interacting Flowfield

Since the position of the Mach disk is unknown a priori, it must be determined by an iterative fashion. It is stipulated that the correct location and configuration of the Mach disk are determined from the condition that as a result of subsequent interaction between the waves and the flowfields, the equivalent one-dimensional analysis for the central core flow will give a sonic condition at a section of minimum area downstream of the triple point.

As the calculations of the flowfield including the coalescence of waves are carried out, the possible occurrence of a Mach configuration along the main incident wave is checked at each step by examining the flow conditions within its neighborhood. Insertion of the Mach configuration is allowed only when triple point state can be satisfied. Once the Mach configuration is inserted, the central core flow is approximated by a one-dimensional treatment. The subsequent interaction-relations between the streams are similar to those between the primary and the secondary streams of a supersonic ejector system[7] and will not be repeated here.[1]

Results and Discussion

The preceding analysis presented for the underexpanded axisymmetric nozzle flow is programed on an IBM system 360/75 digital computer using double precision mode. A single successful run of the Fortran IV program to produce the final correct flowfield including the Mach configuration takes less than 6 min computation time.

Figure 2 illustrates the results of calculation in determining the Mach disk location for the case of $M_e = 1.5$ and $p_e/p_a = 10.0$ with $N = 30$, where N denotes the number of uniformly spaced waves initiated from the initial Mach line. An early insertion of the Mach disk, for example, at $x = 7.022$, will cause a subsonic state ($M = 0.4$) at the throat in the downstream flowfield. On the other hand, the delayed insertion of the Mach disk at $x = 7.616$ will cause the central core flow reaching a sonic condition before a minimum flow area occurs. The correct location of the Mach disk for this flow situation stands at $x \simeq 7.525$. Because of the saddle point characters of the flow at the throat, the downstream flowfield is very sensitive to the location of the Mach disk; a slight change in its location will result in a tremendously different

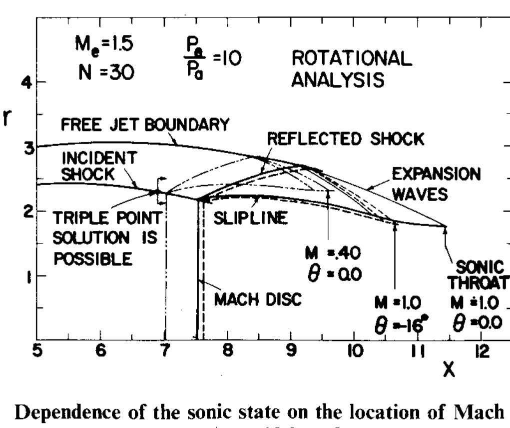

Fig. 2 Dependence of the sonic state on the location of Mach disk at $M_e = 1.5$, $p_e/p_a = 10.0$, and $N = 30$.

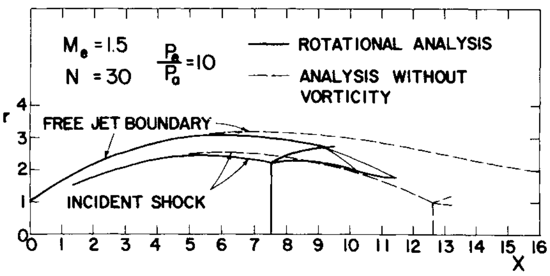

Fig. 3 Comparison of analysis with and without vorticity at $M_e = 1.5$, $p_e/p_a = 10.0$, and $N = 30$.

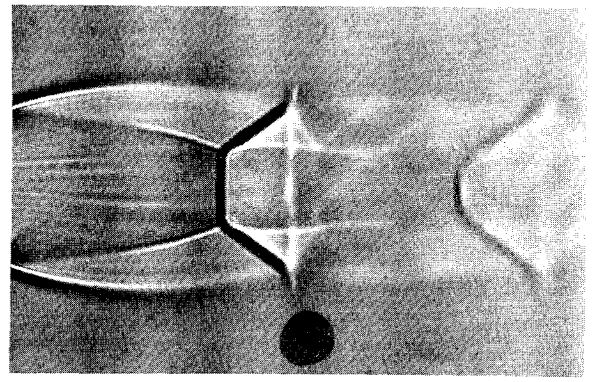

Fig. 5 Shadowgraph photograph of a freejet produced from a convergent nozzle ($p_e/p_a \doteq 2.59$).

downstream flow pattern. The location of the incident shock downstream of which the triple point solution can possibly be found is also shown in the figure.

It has been mentioned previously that because of the enormous difference in stagnation pressure behind the curved incident and the reflected shock waves, the vorticity cannot be neglected in this study. Figure 3 gives a comparison of the numerical results of the flows with and without vorticity for the case of $M_e = 1.5$, $p_e/p_a = 10.0$, and $N = 30$. These results clearly demonstrate the inadequacy of the treatment disregarding the vorticity which gives a much smaller Mach disk standing farther downstream from the nozzle exit. In all of the following results, vorticity has been included in the calculations.

A typical characteristic and shock wave pattern from the actual computation of $M_e = 1.5$, $p_e/p_a = 10.0$, and $N = 30$ is presented in physical scale in Fig. 4. The location and strength of the imbedded incident shock resulting from the coalescence of the compression waves are computed automatically whenever waves intersection occurs. Shown in the same figure are the distributions of various flow properties. The stagnation pressure behind the incident shock drops continuously to a very small value near the triple point. Immediately behind the reflected shock, however, the stagnation pressure becomes higher along the reflected shock and deviates tremendously from the constant stagnation pressure on the upper side of the slipline. This wide variation of stagnation pressure further substantiates the fact that the vorticity cannot be ignored in the calculations. The Mach numbers of the flow ahead and behind the incident and the reflected shocks as well as that above and below the slipline for this particular flow case are also plotted in the same figure.

A shadowgraph photograph of a freejet produced for a convergent nozzle is reproduced in Fig. 5. The remarkable resemblance of the jet configurations and shock patterns as shown in Figs. 4 and 5 suggests that the actual flow processes have been adequately described in the present theoretical considerations.

Some results of calculations for the freejet boundaries and shock wave patterns for $M_e = 1.5$ at different pressure ratios are presented in Fig. 6. It is evident that for a given nozzle Mach number, higher pressure ratios (p_e/p_a) would result in larger Mach disks at farther downstream locations. At the same pressure ratio, however, a reduced nozzle Mach number brings Mach disk closer to the nozzle exit plane.

For lower pressure ratios (e.g., $p_e/p_a = 3.0$ in Fig. 6), some of the calculations produced a somewhat unexpected phenomenon. When the sonic state is reached in the central core, the streamline angle along the slipline does not vanish and thus the area is not at a minimum section. Inserting the Mach disk farther downstream will result in a steeper flow angle at sonic condition. On the other hand, no triple-point solution is possible for early occurrence of the Mach disk. Presumably, this is a result of inaccuracy of the upstream numerical calculation. It has been observed that the flow conditions near the triple point depend on the incident shock strength in a very sensitive manner. The linear interpolation scheme adopted in the calculations of the characteristic grid is probably not accurate enough for cases of lower pressure ratios (p_e/p_a) to account for the rapid change in shock strength near the triple point as well as the effect of the vorticity generation. It is thus necessary to consider that the results presented for lower pressure ratios are interpreted as the upper limit of the location of the Mach disk.

The location and radius of the Mach disk for various nozzle Mach numbers and pressure ratios are presented in Fig. 7. Some experimental data obtained by Love et al.[3] are shown in the same figure for comparison. Good agreement‡ between the results has been observed especially for small nozzle Mach numbers. It is interesting to observe that the curves indicating the influence of the pressure ratio on the size of the Mach disk for different nozzle Mach numbers are crossing each other. It may be con-

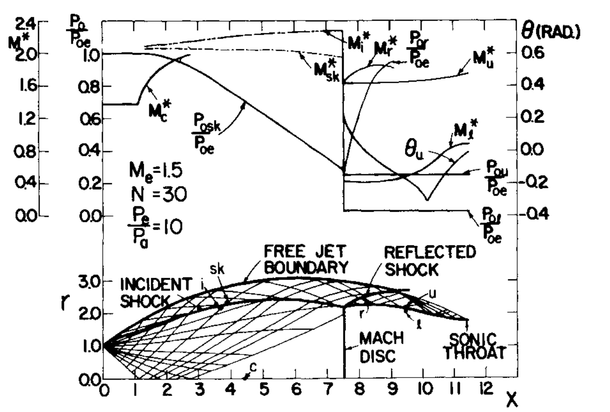

Fig. 4 Characteristic wave pattern in physical scale and the distributions of various flow properties at $M_e = 1.5$, $p_e/p_a = 10.0$, and $N = 30$.

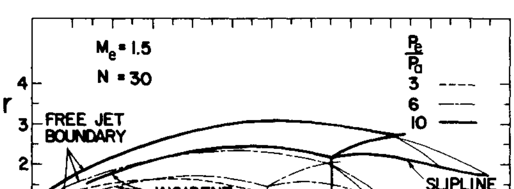

Fig. 6 Freejet boundaries and shock wave patterns at different pressure ratios for $M_e = 1.5$.

‡ Since the submission of this manuscript, a publication by Fox[14] has also appeared. Good agreement has also been obtained in his calculations.

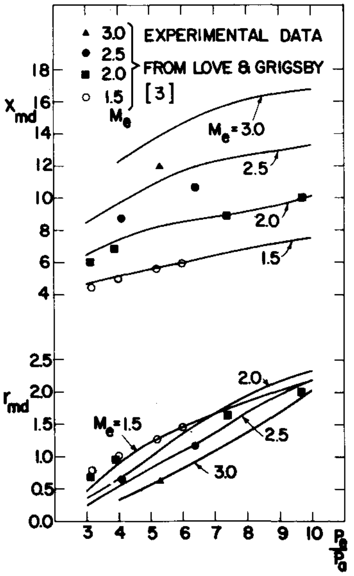

Fig. 7 Location and size of Mach disk, X_{md} and r_{md}, vs p_e/p_a.

ceived that the nonlinear phenomenon pertaining to the Prandtl-Meyer expansion yields larger sizes of the Mach disk at low pressure ratios for smaller nozzle Mach numbers. On the other hand, for larger pressure ratios, larger nozzle Mach numbers tend to maintain the large size of the plumes thereby yielding larger Mach disks at farther downstream positions. Other results pertaining to the maximum size of the freejets and their locations have also compared favorably with the results of Love et al.[1] Additional calculations indicating the influence of the different grid sizes for numerical calculations have also been performed and reported in Ref. 1.

It may be argued that in actual flow situations, the central core flow behind the Mach disk is never uniform and cannot be approximated by a simple one-dimensional analysis. In fact, experimental evidence produced by Back and Cuffel[15] shows that the sonic state extends over a considerable distance downstream of the Mach disk. This apparent discrepancy was observed in early study of the interaction with supersonic ejector systems. For more detailed study of such flowfields,[16] it was found that the one-dimensional choking phenomenon is usually replaced by imposing simultaneously vanishing numerator and denominator of a certain derivative expression pertaining to the flow property of the fluid at the "singular" point. It is known, however, that the one-dimensional throat behaves as a saddle point singularity of the flow so that a slight change of the upstream flow would result in widely different flow conditions near the throat. For the present problem, it is therefore expedient to adopt such a simpler scheme to locate the position and to determine the size of the Mach disk.

Conclusion

It may be concluded from these results of calculations that the appearance of Mach disk within axisymmetric freejet flowfields may be explained from the inviscid interaction between the streams. In addition, the effect of vorticity within these regions is so important that the generation of vorticity due to curved shocks must be evaluated accurately.

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